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Diffractals

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Abstract. Diffractals are waves that have encountered fractals. Fractals are geometric objects with non-integral Hausdorff–Besicovitch dimension D; they have structure down to arbitrarily fine scales. Diffractals are a new wave régime characterised by a short-wave limit in which ever finer levels of structure are explored and geometrical optics is never applicable.

Ths diffractal studied here is the wave $\psi(x, z)$ at distance z beyond a one-dimensional random phase screen that deforms an initially plane wavefront at $z = 0$ into a random fractal curve $h(x)$ with power law spectrum and dimension D (between 1 and 2), after which the wave propagates freely. Some averages of ψ are calculated. These are $\langle\psi\rangle$, $\langle\psi(x, z)\psi^*(x + X, z)\rangle$, the spectrum of ψ and the spectrum of the intensity fluctuations and, most important, the second intensity moment $I_2 \equiv \langle|\psi|^4\rangle$. It is proved that the intensity fluctuations are non-Gaussian. A variety of scaling laws is derived, all involving D. For the 'Brownian' diffractal ($D = 1{\cdot}5$) all averages are expressed exactly in closed form.

$I_2(z)$ varies from 1 (no fluctuations) to 2 (saturated Gaussian fluctuations) as z increases from 0 to ∞. Near $z = 0$, $I_2 \approx 1 + Az^{2-D}$. Near $z = \infty$, $I_2 \approx 2 - B/z$ if $1{\cdot}5 \leq D < 2$ and $I_2 \approx 2 + \Sigma_{n=1}^{\mathrm{int}(2D-2)^{-1}} C_n z^{-2n(D-1)}$ if $1 < D < 1{\cdot}5$, where A, B and C_n are positive constants. Therefore I_2 has a maximum at some value of z if $1 < D < 1{\cdot}5$. For the marginal diffractal $(D \to 1)$ there is an accumulation of power-law decays for large z, giving $I_2 \approx 2 + 1/\ln[(D-1)z]$.

1. Introduction

Diffractals are waves that have encountered fractals. Fractals are geometric objects with regular or random hierarchial structure down to arbitrarily small scales, leading to self-similarity under magnification; in an important study, Mandelbrot (1977) discusses the properties of fractals and shows how they provide an apt description of many of Nature's forms (coastlines, landscapes, blood vessels, rivers, trees, clouds, turbulence, etc) where conventional geometry is inappropriate. In mathematical terms, fractals have a Hausdorff–Besicovitch dimension D that need not be an integer. D refers to the measure of the fractal, considered as a set of points, so that a coastline, for example, with infinitely more points than a smooth curve and infinitely fewer than a finite area, has $1 < D < 2$.

My purpose here is to point out that fractals cause waves to adopt unfamiliar forms that should be studied in their own right, and to examine a particular case in some detail. To see how diffractals differ from more familiar wave fields, consider the limit where the wavelength λ tends to zero. For smooth diffracting objects, the techniques of geometrical optics (Keller 1958) become applicable once λ gets smaller than the smallest length scale, and wave fields are dominated by catastrophes (Thom 1975, Poston and Stewart 1978) on which focusing occurs (Berry 1976). For fractals, no such smallest scale exists, and there is no geometrical optics limit as $\lambda \to 0$. Therefore diffractals constitute a new regime in wave physics.

Diffractals are of wide potential applicability, even for the case considered here where the waves are monochromatic. In practice the mathematical limiting process $\lambda \to 0$ cannot be performed, and what matters is whether the diffracting object has structure on scales near the actual value of λ employed in an experiment; structure on scales much smaller than λ does not affect the wave. If there is no structure on scales near λ, the object can be considered smooth, and conventional short-wave theory is applicable. If there is self-similar structure on a range of scales that includes λ, then the object can be modelled by a fractal and the wave is a diffractal. Examples are: sound and radar diffracted by trees; radio waves scattered by ionospheric turbulence or reflected by landscapes; light and sound in matter near critical points; and quantum waves in classically non-integrable systems.

The simplest diffractal problem, and the only one I shall study in this paper, appears to be to determine the statistics of the wave obtained by taking a plane wave with wavenumber $k(\equiv 2\pi/\lambda)$, propagating in the z direction, namely

$$\psi(x, z) = \mathrm{e}^{\mathrm{i}kz} \qquad (z<0), \tag{1.1}$$

and imposing on it a random fractal phase modulation $kh(x)$ at $z=0$, so that the wave just beyond $z=0$ becomes

$$\psi(x, 0^+) = \mathrm{e}^{\mathrm{i}kh(x)}. \tag{1.2}$$

Geometrically, this modulation amounts to taking plane wavefronts near $z=0$ and deforming them into random fractals with shape $z=-h(x)$. Physically, it approximates a wave reflected by a fractal surface or refracted by a slab of transparent material with fractally turbulent refractive index. $h(x)$ is a Gaussian random fractal function with power-law spectrum and dimension D lying between 1 and 2, whose properties will be described in § 2.

For $z>0$ the wave propagates freely and $\psi(x, z)$, the function whose statistics are to be determined, is given by an elementary diffraction integral, stated in § 3. Also in § 3 are formulae for some simple statistics of ψ: the average wave $\langle\psi\rangle$, the coherence function $\langle\psi(x, z)\psi^*(x+X, z)\rangle$, and the power spectrum of ψ (angle brackets denote an average over the ensemble of fractals $h(x)$).

The more interesting statistics, however, are those characterising the intensity fluctuations, and the most important of these is the second moment.

$$I_2(z) \equiv \langle I^2(x, z)\rangle, \tag{1.3}$$

where

$$I(x, z) \equiv |\psi(x, z)|^2 \tag{1.4}$$

is the wave intensity. In § 3 an integral expression for I_2 is derived; it involves only the fractal dimension D of $h(x)$, and one other dimensionless parameter incorporating z. This result is the simplest of the scaling laws expected to pervade diffractal theory. Also in § 3 is an integral for the spectrum of intensity fluctuations, namely

$$P_I(K, z) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}X\, \langle (I(x+X, z)-\langle I\rangle)(I(x, z)-\langle I\rangle)\rangle\, \mathrm{e}^{\mathrm{i}KX}, \tag{1.5}$$

and a discussion of its properties.

It is proved in § 4 that ψ cannot be a Gaussian random function, but must fluctuate according to some other statistics (still unknown).

There is one special case for which I_2 and P_I can be evaluated exactly in closed form. This is when the initial wavefront is a 'Brownian' fractal with $D = 1\cdot 5$, and is discussed in § 5.

Sections 6 and 7 are devoted to expansions of I_2 for small and large z respectively.

In the case of diffraction from a smooth phase screen it is well known that I_2 can rise to very large values for short waves, because of focusing (Salpeter 1967, Shishov 1971, Buckley 1971, Jakeman and McWhirter 1977, Uscinski 1977), and Berry (1977) showed that the higher moments $\langle I^n\rangle$ diverge in a manner governed by the whole hierarchy of focusing catastrophes. Such large values of I_2 do not occur for diffractals (there are no rays to be focused), and this means that the approach to the marginal diffractal $D = 1$, which separates fractal from smooth phase screens, is highly singular. This limit is discussed in § 8.

The analysis in §§ 7 and 8 is complicated and bears out the expectation that diffractals involve functions new to wave theory.

Recently, and without employing fractal terminology, Gochelashvily and Shishov (1975), Rumsey (1975), Marians (1975) and Furuhama (1975) have considered diffraction from self-similar phase screens. My results in this paper extend and complement theirs.

2. Fractal phase screen

The shape $h(x)$ of the wavefront at $z = 0^+$ is a Gaussian random function of x (Rice 1944, 1945). This means that the average of any function involving h can be found after Fourier transformation and using the relation

$$\langle e^{iG(h(x))}\rangle = e^{i\langle G\rangle}\, e^{-(\langle G^2\rangle - \langle G\rangle^2)/2}, \tag{2.1}$$

where G is any linear functional of h. To find $\langle G\rangle$ and $\langle G^2\rangle$, the conventional procedure is to specify the mean $\langle h\rangle$ and the correlation $\langle h(x_1)h(x_2)\rangle$. But the $h(x)$ considered here is a fractal function, for which the correlation and hence the variance $\langle h^2\rangle$ is infinite, a fact which follows from the invariance of the graph of h under magnification. Therefore we specify h by its spectrum $\bar{h}(K)$ which we take as the power law

$$\bar{h}(K) = A/|K|^\alpha \qquad (1 < \alpha < 3); \tag{2.2}$$

the meanings of the constant A and the exponent α will soon be apparent.

It follows from (2.2) with the stated range of α that the correlation of h (which is simply the Fourier transform of $\bar{h}(K)$), and in particular $\langle h^2\rangle$, is infinite as asserted. But the diffractal statistics (§ 3) will turn out to depend on the mean-square increment of $h(x)$ in a distance X, and this is finite, being given by

$$\langle (h(x+X) - h(x))^2\rangle = \int_{-\infty}^{\infty} dK\, \bar{h}(K)(e^{iKX} - 1) = \frac{2A}{\alpha - 1}\sin\frac{\pi}{2}(2-\alpha)\Gamma(2-\alpha)|X|^{\alpha-1}, \tag{2.3}$$

which results from use of (2.2) and integration by parts. It also follows from (2.2) that the mean-square slope of h, given by the Fourier transform of $K^2\bar{h}(K)$, is infinite, and the divergence arises for large K, so that $h(x)$ is non-differentiable (like all fractals). However, the chords joining h-values separated by a distance X do have finite mean-square slope (from (2.3)), and this leads to a definition of the strength of the fractal that is more easily interpreted than the constant A, namely the distance is over which the chord has an r.m.s. slope of one radian. Thus

$$\langle (h(x+L)-h(x))^2\rangle/L^2 \equiv 1, \tag{2.4}$$

so that A can be identified and (2.3) replaced by

$$\langle (h(x+X)-h(x))^2\rangle = L^{3-\alpha}|X|^{\alpha-1}. \tag{2.5}$$

I shall follow Sayles and Thomas (1978, see also Berry and Hannay 1978) and call L the 'topothesy' of $h(x)$.

The exponent α is related to the fractal dimension D of the graph of $h(x)$ by

$$D=(5-\alpha)/2. \tag{2.6}$$

This is proved in Appendix 1, which is included here to make the original argument of Orey (1970) more accessible. Thus D varies from 1 to 2 as α varies from 3 to 1. According to (2.5), $h(x)$ obeys the following similarity law: if the graph of $h(x)$ is stretched along the x axis by a factor f, and along the h axis by a factor f^{2-D}, the resulting curve is statistically indistinguishable from the original. This means not only that h has no smallest scale, as implied by its being a fractal, but also that h has no largest scale, and is correlated with itself (equation (2.5)) over arbitrarily large distances. Pictures of such scale-free fractal functions of one variable (as opposed to other fractals amply illustrated by Mandelbrot (1977)) will be published by Berry and Lewis (1979).

Three special values of D are of particular interest. $D=1$ is the *marginal fractal*, where $h(x)$ is 'almost' a smooth function. The integral leading to (2.3) diverges and L disappears from (2.5), but it is still possible to define a particular scale-free function by

$$\langle (h(x+X)-h(x))^2\rangle = \beta^2X^2, \tag{2.7}$$

where β gives the r.m.s. 'slope' of chords of any length and replaces L as a measure of the strength of h. Essentially this marginal fractal was defined by Nye (1970) for application to glaciology. $D=1{\cdot}5$ is the *Brownian fractal*, because the r.m.s. increment of h then varies like $X^{1/2}$, as for one-dimensional Brownian motion along the h axis in time X. $D=2$ is the *extreme fractal*, where the graph of h is on the verge of filling a finite area; it corresponds to the so-called '$1/f$ noise' for functions of time.

3. Diffractal integrals and scaling laws

The simplest way to solve the diffraction problem posed in § 1 is to Fourier-analyse the boundary condition (1.2) and fit it to ψ expressed as an angular spectrum of plane waves moving towards $z=+\infty$ (Uscinski 1977). It greatly simplifies all subsequent formulae, and introduces no essential loss of generality, to make the paraxial approximation, according to which the only significant plane-wave components of ψ are those travelling in directions making small angles with the z axis. Considering that it is customary to justify paraxiality by restricting attention to wavefronts $h(x)$ with small slopes, it is by no means obvious that any diffractal, arising as it does from a non-differentiable wavefront, can be paraxial. In fact it will emerge presently that ψ is in fact paraxial whenever the topothesy L (equation (2.5)) is small enough. Under this assumption ψ is given by

$$\psi(x, z) = e^{i(kz-\pi/4)}\left(\frac{k}{2\pi z}\right)^{1/2} \int_{-\infty}^{\infty} dx' \, e^{ik[h(x')+(x-x')^2/2z]}. \tag{3.1}$$

The simplest average is $\langle\psi\rangle$, and (2.1) and (3.1) give

$$\langle\psi\rangle = e^{ikz} \, e^{-k^2\langle h^2\rangle/2} = 0, \tag{3.2}$$

since $\langle h^2\rangle$ is infinite for scalefree fractals.

Next comes the lowest-order coherence function $\langle\psi(x, z)\psi^*(x+\xi, z)\rangle$, given by

$$\begin{aligned}\langle\psi(x, z)\psi^*(x+\xi, z)\rangle &= \frac{k}{2\pi z}\int dx' \int dx'' \, \langle e^{ik(h(x')-h(x''))}\rangle \, e^{ik[(x-x')^2-(x+\xi-x'')^2]/2z} \\ &= e^{-k^2L^{2(D-1)}|\xi|^{2(2-D)}/2},\end{aligned} \tag{3.3}$$

an expression that follows easily from (2.1) and (2.5). This result is independent of z, as expected (Booker *et al* 1950). The $\xi = 0$ limit, namely

$$\langle|\psi|^2\rangle = \langle I\rangle = 1, \tag{3.4}$$

follows from the conservation of energy together with (1.1).

The power spectrum $P_\psi(K)$ of ψ is simply the Fourier transform of (3.3), and is a function peaked about $K = 0$, with the following limiting behaviour:

$$\begin{aligned}kP_\psi(K) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} du \, e^{iKu/k} \, e^{-(kL)^{2(D-1)}|u|^{2(2-D)}/2} \\ &\to \begin{cases} 2^{1/2(2-D)}\Gamma[(5-2D)/(4-2D)]/\pi(kL)^{(D-1)/(2-D)} & (|K|\to 0) \\ (kL)^{2(D-1)}\sin[\pi(2-D)]\Gamma(5-2D)/2\pi|K/k|^{5-2D} & (|K|\to\infty)\end{cases}\end{aligned} \tag{3.5}$$

From these results it is possible to get a conditions for the validity of the paraxial approximation, based on the requirement that $P_\psi(K)/P_\psi(0) \ll 1$ when the scattering angle $K/k \gg Q$, where $Q \ll 1$. From (3.5),

$$Q = (kL)^{(D-1)/(2-D)}, \tag{3.6}$$

confirming that all significant diffracted waves are paraxial if L is small enough. Notice that Q increases as λ gets smaller, showing the effect of the initial wavefront's increasing 'slope', as measured by chords joining points λ apart (equation (2.5)).

The large-$|K|$ limit in (3.5) fails for the marginal diffractal $D = 1$. In fact the coefficients of all inverse powers of $|K|$ in the asymptotic expansion of $P_\psi(K)$ vanish as $D \to 1$, because P_ψ is then exponentially small as $|K| \to \infty$. For this case, use of (2.7) gives

$$P_\psi(K) = (1/k\sqrt{2\pi}) \, e^{-K^2/2k^2\beta^2} \qquad (D = 1), \tag{3.7}$$

implying paraxiality when the 'slope' $\beta \ll 1$, as expected in this case where the initial wavefront is 'almost' smooth.

The most interesting statistics, however, are those characterising the intensity fluctuations, and the simplest of these is the second moment $I_2(z)$ defined by (1.3).

Direct averaging of $|\psi|^4$ using (3.1) and (2.5) involves

$$\langle (h(x_1)+h(x_2)-h(x_3)-h(x_4))^2\rangle$$
$$= L^{2(D-1)}(|x_1-x_3|^{4-2D}+|x_2-x_4|^{4-2D}+|x_2-x_3|^{4-2D}$$
$$+|x_1-x_4|^{4-2D}-|x_1-x_2|^{4-2D}-|x_3-x_4|^{4-2D}). \tag{3.8}$$

I_2 is an integral over x_1, x_2, x_3 and x_4; a linear change of variables makes two integrations easy and leads to

$$I_2(z)=\frac{k}{2\pi z}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\;\mathrm{e}^{\mathrm{i}kxy/z}\,\mathrm{e}^{-k^2L^{2(D-1)}(2|x|^{4-2D}+2|y|^{4-2D}-|x+y|^{4-2D}-|x-y|^{4-2D})/2}. \tag{3.9}$$

Introducing the dimensionless parameter

$$\zeta \equiv kz(kL)^{(D-1)/(2-D)}/2^{1/(4-2D)}=kzQ/2^{1/(4-2D)}, \tag{3.10}$$

where Q is the scattering angle defined by (3.6), and making full use of the symmetry of the integrand in (3.9) gives

$$I_2(\zeta)=\frac{4}{\pi\zeta}\int_0^{\infty}\mathrm{d}u\int_u^{\infty}\mathrm{d}v\,\cos\frac{uv}{\zeta}\,\mathrm{e}^{-[2u^{4-2D}+2v^{4-2D}-(v+u)^{4-2D}-(v-u)^{4-2D}]}. \tag{3.11}$$

This integral will be the principal object of study from now on. Like all statistics of the intensity $|\psi|^2$ it involves only the mean-square increment of h and so is well-defined even though $\langle h^2\rangle$ is infinite. As will become apparent, the limiting values of I_2 are

$$I_2(\zeta)\to\begin{cases}1 & (\zeta\to 0)\\ 2 & (\zeta\to\infty)\end{cases}. \tag{3.12}$$

This result is not restricted to diffractals, but holds also for smooth phase screens. For intermediate values of ζ the most important question is: does $I_2(\zeta)$ possess a maximum like the focusing peak of strength $\mathrm{O}(\ln k)$ that occurs for smooth phase screens (Shishov 1971, Buckley 1971), or does it rise monotonically from 1 to 2 as ζ increases? The answer (§ 7) will be that I_2 has a weak maximum for the 'less rough' fractals with $D<1{\cdot}5$; of course this regime of more intense fluctuations is not caused by focusing of rays, since fractal wavefronts, being non-differentiable, have no normals.

The parameter ζ (equation (3.10)) embodies a scaling law, showing how the same change in intensity fluctuations can be produced by increasing the topothesy, moving farther from the screen, or diminishing the wavelength.

The spectrum of intensity fluctuations $P_I(K,z)$, defined by (1.5), can be expressed as a single integral using arguments like those leading to (3.11):

$$kP_I(K,z)=\frac{2^{1/(4-2D)}}{\pi Q}\int_0^{\infty}\mathrm{d}u\,\cos\!\left(2^{1/(4-2D)}\frac{KQu}{k}\right)$$
$$\times\{\mathrm{e}^{-[2u^{4-2D}+2(K\zeta/k)^{4-2D}-(u+K\zeta/k)^{4-2D}-|u-K\zeta/k|^{4-2D}]}-1\} \tag{3.13}$$

This has the limits

$$kP_I(K,z)\to\begin{cases}0 & (\zeta\to 0)\\ 2^{-1/(4-2D)}kP_\psi(2^{-1/(4-2D)}K) & (\zeta\to\infty)\end{cases}, \tag{3.14}$$

where P_ψ is given by (3.5). Thus the intensity spectrum saturates far from the screen and becomes a scaled version of the spectrum of ψ itself. This result was obtained by approximating the exponent in the first term in braces in (3.13) for $K\zeta/k \gg 0$. But this leaves the term -1 in braces, giving a contribution to P_I proportional to $-\delta(K)$. However, a careful analysis shows this to be cancelled by the residual region $u \gg K\zeta/k$ in the integration of the first term.

Equation (3.13) shows that the spectrum obeys the following general scaling law:

$$P_I(K/k, Q, \zeta) = P_I(K/kQ, 1, \zeta Q)/Q. \tag{3.15}$$

Therefore when the wavefront deformation (as measured by Q) is larger, features of the spectrum occur at larger angles and closer to the screen.

4. Diffractals are non-Gaussian

It is not known what probability distribution the diffractal intensity I possesses. However, a consideration of the second moment I_2 is sufficient to rule out the possibility that the wave ψ obeys Gaussian statistics, except far from the screen. To show this let

$$\psi(x, z) = \xi(x, z) + \mathrm{i}\eta(x, z), \tag{4.1}$$

pretend that the real functions ξ and η are Gaussian random functions of x, and evaluate

$$I_2 = \langle \zeta^4 \rangle + \langle \eta^4 \rangle + 2\langle \xi^2 \eta^2 \rangle. \tag{4.2}$$

Use of (2.1) and (3.2) gives

$$\langle \xi^4 \rangle = \frac{\mathrm{d}^4}{\mathrm{d}\alpha^4} \langle \mathrm{e}^{\mathrm{i}\alpha\xi} \rangle |_{\alpha=0} = 3\langle \xi^2 \rangle^2, \tag{4.3}$$

and

$$\langle \xi^2 \eta^2 \rangle = \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} \frac{\mathrm{d}^2}{\mathrm{d}\beta^2} \langle \mathrm{e}^{\mathrm{i}(\alpha\xi+\beta\eta)} \rangle |_{\alpha=0} = \langle \xi^2 \rangle \langle \eta^2 \rangle + 2\langle \xi\eta \rangle^2, \tag{4.4}$$

so that

$$I_2 = 3\langle \xi^2 \rangle^2 + 3\langle \eta^2 \rangle^2 + 2\langle \xi^2 \rangle \langle \eta^2 \rangle + 2\langle \xi\eta \rangle^2. \tag{4.5}$$

To find $\langle \xi^2 \rangle$, $\langle \eta^2 \rangle$ and $\langle \xi\eta \rangle$ consider

$$\langle \psi^2 \rangle = \langle \xi^2 \rangle - \langle \eta^2 \rangle + 2\mathrm{i}\langle \xi\eta \rangle. \tag{4.6}$$

For the diffractal (3.1) this depends on $\langle \exp[\mathrm{i}(h(x_1) + h(x_2))] \rangle$, which vanishes on account of (2.1), and the fact that $\langle h^2 \rangle$ is infinite. Therefore

$$\langle \xi\eta \rangle = 0, \qquad \langle \xi^2 \rangle = \langle \eta^2 \rangle = \tfrac{1}{2}, \tag{4.7}$$

the last equality following from (3.4).

Thus, if diffractals were Gaussian, the second moment, from (4.5), would be

$$I_2 = 2. \tag{4.8}$$

This is a special case of the more general relationship

$$\langle I(x)I(x+X) \rangle - \langle I \rangle^2 = \langle \psi(x)\psi^*(x+X) \rangle^2, \tag{4.9}$$

which holds for Gaussian waves with vanishing $\langle \psi \rangle$ and $\langle \psi^2 \rangle$. But I_2 is given by (3.11) and only equals 2 as $\zeta \to \infty$ (and possibly at other isolated values depending on D). Therefore, in general, diffractals are not Gaussian random waves.

The exceptional case is far from the screen. Of course the reasoning based on (4.8) does not prove that ψ must be Gaussian at $\zeta = \infty$, only that it may be. Moreover, the standard argument (Mercier 1962), which depends on the asymptotic independence of parts of $h(x)$ separated by large distances, does not apply to fractal phase screens. Nevertheless, it seems likely that the conclusion still holds, and ψ does become Gaussian far from the screen. Corroboration of this opinion is given by the asymptotic correlations (3.14), which with the aid of (3.5) can be shown to satisfy the Gaussian relation (4.9).

5. Brownian diffractals

For the wavefront with dimension $D = 1{\cdot}5$ the diffractal formulae of § 3 can be evaluated in closed form using elementary methods. The second moment I_2 (equation (3.11)) is

$$I_2(\zeta) = 2\{1 - [\tfrac{1}{2} - C((2\zeta/\pi)^{1/2})]^2 - [\tfrac{1}{2} - S((2\zeta/\pi)^{1/2})]^2\}, \tag{5.1}$$

where C and S are Fresnel integrals defined by

$$C(t) + \mathrm{i}S(t) \equiv \int_0^t \mathrm{d}x \, \mathrm{e}^{\mathrm{i}\pi x^2/2}. \tag{5.2}$$

This result, exact for the present case, was obtained by Jakeman and McWhirter (1977) as an approximation to a slightly different case; their figure 6 shows $I_2 - 1$ rising monotonically from 0 to 1 as ζ increases. The limiting behaviour of (5.1) is

$$I_2(\zeta) \to \begin{cases} 1 + 2(2\zeta/\pi)^{1/2} & (\zeta \to 0) \\ 2 - 1/\pi\zeta & (\zeta \to \infty). \end{cases} \tag{5.3}$$

The power spectrum of ψ (equation (3.5)) is

$$P_\psi(K) = 2L/\pi(k^2L^2 + 4K^2/k^2), \tag{5.4}$$

and the power spectrum of intensity fluctuations (equation (3.13)) is

$$P_I(K, z) = \frac{L}{\pi(k^2L^2 + K^2/k^2)}\left[1 - \mathrm{e}^{-LzkK}\left(\frac{Lk^2}{K}\sin\frac{zk^2}{k} + \cos\frac{zk^2}{k}\right)\right], \tag{5.5}$$

which satisfies (3.14).

6. Close to the screen

To find the leading terms in $I_2(\zeta)$ when ζ is small it is convenient to change variables in (3.11) by $u \to u\sqrt{\zeta}$, $v \to v\sqrt{\zeta}$ and expand the exponential in powers of ζ. Employing the full u, v plane as in (3.9) and making use of the symmetry of the integrand gives

$$I_2(\zeta) = \frac{\mathrm{Re}}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}v \, \mathrm{e}^{\mathrm{i}uv}[1 + \zeta^{2-D}(-4|u|^{4-2D} + 2|u+v|^{4-2D})] + \mathrm{O}(\zeta^{4-2D}). \tag{6.1}$$

The first term in square brackets gives unity. The second term is the integral of $|u|^{4-2D}\delta(u)$ and hence zero. The third term becomes

$$\frac{\mathrm{Re}}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}v\, \mathrm{e}^{\mathrm{i}uv}|u+v|^{4-2D}$$

$$= \frac{\mathrm{Re}\, 2^{4-2D}}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}x^2}|x|^{4-2D}\int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}y^2/4}$$

$$= \frac{2^{5-2D}}{\sqrt{\pi}}\mathrm{Re}\, \mathrm{e}^{-\mathrm{i}\pi/4}\int_{0}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}x^2}|x|^{4-2D}$$

$$= 2^{4-2D}\cos\frac{\pi}{2}(2-D)\Gamma\left(\frac{5-2D}{2}\right)\Big/\sqrt{\pi}. \tag{6.2}$$

Therefore the second moment close to the screen is

$$I_2(\zeta) = 1 + 2^{5-2D}\zeta^{2-D}\cos\frac{\pi}{2}(2-D)\Gamma\left(\frac{5-2D}{2}\right)\Big/\sqrt{\pi} + \mathrm{O}(\zeta^{4-2D}). \tag{6.3}$$

In the Brownian case ($D = 1{\cdot}5$) this result agrees with (5.3). In the extreme case ($D \to 2$) I_2 grows away from unity very quickly, suggesting a rapid approach to the far-field Gaussian regime where $I_2 = 2$. As the marginal case is approached and $D = 1+\epsilon$, where $\epsilon \to 0$, I_2 is given by

$$I_2(\zeta) = 1 + 2\pi\epsilon\zeta + \mathrm{O}(\epsilon^2\zeta^2), \tag{6.4}$$

so that I_2 shows an infinitely slow linear growth. It is interesting to contrast this behaviour with that close to a smooth initial wavefront with correlation

$$\langle h(x)h(x+X)\rangle = H^2\,\mathrm{e}^{-X^2/L^2}, \tag{6.5}$$

For which a short calculation shows that I_2 grows quadratically:

$$I_2(z) = 1 + 12z^2H^2/L^4 + \mathrm{O}(z^3). \tag{6.6}$$

This expression does not involve k and can indeed be alternatively derived from geometrical optics. The fact that (6.4) and (6.6) depend differently on z indicates that the marginal fractal is not smooth enough to have normals that behave like rays.

7. Far from the screen

The purpose of this section is to study how the intensity fluctuations approach saturation at the Gaussian value $I_2 = 2$ as $\zeta \to \infty$. If I_2 approaches 2 from above, then (3.12) shows that $I_2(\zeta)$ must possess a maximum for some value of ζ, as for short-wave diffraction from a smooth initial wavefront. If I_2 approaches 2 from below, then $I_2(\zeta)$ need have no maximum, and I shall give an argument strongly suggesting that indeed there is no maximum in this case.

Starting from (3.11) the first step is to integrate the cosine term by parts, denoting the upper limit of v by V, which will soon be set equal to infinity. Thus

$$I_2(\zeta) = \frac{4}{\pi}\int_0^{\infty}\frac{\mathrm{d}u}{u}\,\mathrm{e}^{-2u^{4-2D}}\left\{\left[\sin\left(\frac{uv}{\zeta}\right)\mathrm{e}^{-g(u,v)}\right]_u^V + \int_u^{\infty}\mathrm{d}v\,\sin\left(\frac{uv}{\zeta}\right)\frac{\partial g(u,v)}{\partial v}\,\mathrm{e}^{-g(u,v)}\right\}, \tag{7.1}$$

where

$$g(u,v)=2v^{4-2D}-(v+u)^{4-2D}-(v-u)^{4-2D}$$
$$\rightarrow 4(2-D)(D-1\cdot 5)u^2V^{-2(D-1)} \qquad (v \gg 1). \tag{7.2}$$

The upper limit of the first term contributes

$$I_2^{(1)}=\frac{4}{\pi}\int_0^\infty \frac{\mathrm{d}u}{u}\sin\frac{uV}{\zeta}\,\mathrm{e}^{-2u^{4-2D}-6(2-D)(D-1\cdot 5)u^2V^{2(D-1)}}$$
$$=\frac{4}{\pi}\int_0^\infty \mathrm{d}x\frac{\sin x}{x}\,\mathrm{e}^{2x^{4-2D}\zeta^{4-2D}/V^{4-2D}}\,\mathrm{e}^{-6(2-D)(D-1\cdot 5)x^2\zeta^2/V^{2D}}$$
$$\rightarrow\frac{4}{\pi}\int_0^\infty \mathrm{d}x\frac{\sin x}{x}=2 \qquad \text{as } V\rightarrow\infty. \tag{7.3}$$

The lower limit of the first term contributes

$$I_2^{(2)}=\frac{-4}{\pi}\int_0^\infty \frac{\mathrm{d}u}{u}\sin\frac{u^2}{\zeta}\,\mathrm{e}^{-(4-2^{4-2D})u^{4-2D}}$$
$$\rightarrow-\frac{2\Gamma[1/(2-D)]}{\pi\zeta(2-D)(4-2^{4-2D})^{1/(2-D)}} \qquad \text{as } \zeta\rightarrow\infty. \tag{7.4}$$

Thus

$$I_2(\zeta)=2-\frac{2\Gamma[1/(2-D)]}{\pi\zeta(2-D)(4-2^{4-2D})^{1/(2-D)}}+\frac{4}{\pi}\int_0^\infty\frac{\mathrm{d}u}{u}\,\mathrm{e}^{-2u^{4-2D}}J(u,\zeta)+\mathrm{O}(\zeta^{-3}), \tag{7.5}$$

where

$$J(u,\zeta)\equiv 2(2-D)\int_u^\infty \mathrm{d}v\,\sin\frac{uv}{\zeta}(2v^{3-2D}+(v+u)^{3-2D}$$
$$-(v-u)^{3-2D})\,\mathrm{e}^{-(2v^{4-2D}-(v+u)^{4-2D}-(v-u)^{4-2D})}. \tag{7.6}$$

The most obvious way to approximate J for large ζ is to replace $\sin uv/\zeta$ by uv/ζ. However, the limit in (7.2) shows that if this is done the resulting integral for J exists only if $D>1\cdot 5$. In this case $J<0$, and the second moment saturates as

$$I_2(\zeta)\rightarrow 2-A(D)/\zeta \qquad (D>1\cdot 5, \zeta\rightarrow\infty). \tag{7.7}$$

An approximation to the constant A can be obtained when D is close to $1\cdot 5$. Then J is dominated by the contribution from large v in (7.6), and is given by

$$J(u,\zeta)\approx-4(2-D)(D-1)u^{6-2D}/\zeta \qquad (0<D-1\cdot 5\ll 1, \zeta\rightarrow\infty). \tag{7.8}$$

Therefore

$$A(D)\approx\frac{2\Gamma[1/(2-D)]}{\pi(2-D)(4-2^{4-2D})^{1/(2-D)}}+\frac{8(D-1)\Gamma[(3-D)/(2-D)]}{\pi 2^{(3-D)/(2-D)}}. \tag{7.9}$$

This procedure fails for the 'smoother-than-Brownian' fractals $D<1\cdot 5$, and the sine factor in (7.6) must be retained in order to ensure convergence, implying that I_2 approaches 2 more slowly than ζ^{-1}. For large ζ the main contribution to J comes from large v, so that the limit (7.2) will again be used. However, it is necessary to retain the exponential in (7.6) to be sure of keeping all terms in J that decay slower than ζ^{-1}. This gives

$$J(u,\zeta)\approx 8(2-D)(1\cdot 5-D)(D-1)u^2\int_u^\infty \mathrm{d}v\,\sin\frac{uv}{\zeta}v^{1-2D}$$

$$\times\sum_{n=0}^{\infty}\frac{[4(2-D)(1\cdot 5-D)u^2v^{-2(D-1)}]^n}{n!}\left(1+\mathrm{O}\left(\frac{u^2}{v^2}\right)\right). \tag{7.10}$$

Analysis of the integrand shows that terms for which

$$n>\mathrm{Int}[(3-2D)/2(D-1)]\equiv N(D), \tag{7.11}$$

Where Int denotes 'the integer part of', converge as $v\to\infty$ if the sine term is replaced by uv/ζ, and the same is true for all correction terms of order u^2/v^2. All these terms give a contribution to I_2 of order ζ^{-1}.

In the remaining terms the range of integration will be split into two for a reason soon to be apparent, giving

$$J(u,\zeta)\approx 2(D-1)\sum_{n=0}^{N(D)}\frac{[4(2-D)(1\cdot 5-D)u^2]^{n+1}}{n!}$$

$$\times\left(\int_0^\infty \mathrm{d}u\,v^{-1}v^{-2(D-1)(n+1)}\sin\frac{uv}{\zeta}-\frac{u}{\zeta}\int_0^u \mathrm{d}v\,v^{-2(D-1)(n+1)}\right)$$

$$=\sum_{n=0}^{N(D)}\frac{\Gamma(1-2(D-1)(n+1))\sin[\pi(D-1)(n+1)]}{(n+1)!}\left(\frac{4(2-D)(1\cdot 5-D)u^{2D}}{\zeta^{2(D-1)}}\right)^{n+1}$$

$$-\sum_{n=0}^{N(D)}\frac{[4(2-D)(1\cdot 5-D)^{n+1}u^{(4-2D)(n+1)+2}}{n!\zeta[1-2(D-1)(n+1)]}+\mathrm{O}(\zeta^{-1}). \tag{7.12}$$

The first group of terms gives the contribution decaying slower than ζ^{-1}. According to (7.11) the number of such terms increases discontinuously as D decreases from 1·5 to 1. For $\frac{3}{2}>D>\frac{5}{4}$ only $n=0$ contributes; for $\frac{5}{4}>D>\frac{7}{6}$ only $n=0$ and $n=1$ contribute, etc. At values of D given by

$$D_n=(3+2n)/2(n+1) \tag{7.13}$$

the nth term begins to contribute. It is then of order ζ^{-1} and its coefficient is infinite. These singular contributions at $D=D_n$ are cancelled by the second group of terms in (7.12), which are all of order ζ^{-1}; it was precisely to produce this cancellation that the integral over v was split into two. The remaining terms labelled $\mathrm{O}(\zeta^{-1})$ in (7.12) are not singular functions of D.

To evaluate I_2, the final step is to perform the integration over u in (7.5), and this gives

$$I_2(\zeta)=2+\frac{2}{\pi(2-D)}\sum_{n=0}^{N(D)}\frac{[4(2-D)(1\cdot 5-D)]^{n+1}}{(n+1)!}$$

$$\times\left\{\frac{\Gamma[1-2(D-1)(n+1)]\sin[\pi(D-1)(n+1)]\Gamma[D(n+1)/(2-D)]}{2^{D(n+1)/(2-D)}\zeta^{2(D-1)(n+1)}}\right.$$

$$\left.-\frac{2(D-1)(n+1)\Gamma[n+1+1/(2-D)]}{2^{n+1+1/(2-D)}\zeta[1-2(D-1)(n+1)]}\right\}$$

$$-\frac{2\Gamma[1/(2-D)]}{\pi(2-D)(4-2^{4-2D})^{1/(2-D)}\zeta}+\mathrm{O}(\zeta^{-1}). \tag{7.14}$$

The leading term in this expression is of order $\zeta^{-2(D-1)}$ and was obtained by Gochelashvily and Shishov (1975) for the case where the initial wavefront is a fractal surface rather than a fractal curve. In the Brownian case ($D = 1{\cdot}5$) the terms in braces cancel, and the remaining explicitly written terms agree precisely with the limit in (5.3).

The results (7.7) and (7.14) show that $I_2(\zeta)$ approaches 2 from below for 'rougher' fractals $2 > D \geqslant 1{\cdot}5$, and from above for 'smoother' fractals $1{\cdot}5 > D > 1$. Therefore $I_2(\zeta)$ has a maximum for $1{\cdot}5 > D > 1$. For the intermediate case $D = 1{\cdot}5$ the formula (5.1) shows that I_2 has no maximum, so it is most unlikely that there is a maximum for $D > 1{\cdot}5$. The maximum that occurs for $D < 1{\cdot}5$ represents a concentration of intensity fluctuations stronger than Gaussian, and is a vestigial form of the strong focusing fluctuations that occur for smooth initial wavefronts. For the rougher fractals no such concentration occurs, and the fluctuations are always weaker than Gaussian.

8. Marginal diffractal

The limiting case $D = 1$ is particularly interesting as it separates diffractals from the physically very different ordinary diffracted random waves. Naively putting $D = 1$ in (3.11) gives $I_2 = 1$ and merely shows that the approach to saturation at $I_2 = 2$ occurs only for infinitely large ζ. A better procedure is to study the approach to the limit by setting

$$D \equiv 1 + \epsilon \tag{8.1}$$

and letting ϵ become small. Then in (3.11) the exponent is

$$\begin{aligned}
&[2u^{4-2D} + 2v^{4-2D} - (v+u)^{4-2D} - (v-u)^{4-2D}] \\
&\quad = 2u^2\,\mathrm{e}^{-2\epsilon\ln u} + 2v^2\,\mathrm{e}^{-2\epsilon\ln v} - (v+u)^2\,\mathrm{e}^{-2\epsilon\ln(v+u)} - (v-u)^2\,\mathrm{e}^{-2\epsilon\ln(v-u)} \\
&\quad \approx -2\epsilon u^2[2(v/u)^2\ln(v/u) - (1+v/u)^2\ln(1+v/u) \\
&\qquad - (v/u-1)^2\ln(v/u-1)] \qquad \text{as } \epsilon \to 0.
\end{aligned} \tag{8.2}$$

On putting $v/u = t$, $u^2 \equiv x\zeta$, (3.11) becomes

$$I_2(\zeta) = 1 + \frac{2}{\pi}\int_0^\infty \mathrm{d}x \int_1^\infty \mathrm{d}t \cos xt\; \mathrm{e}^{-2\epsilon\zeta x\chi(t)}, \tag{8.3}$$

where

$$\chi(t) \equiv (1+t)^2\ln(1+t) + (t-1)^2\ln|t-1| - 2t^2\ln t \to 2\ln t \qquad \text{as } t \to \infty, \tag{8.4}$$

and where the term 1 is included after a careful consideration of the contribution from $x = 0$. The x integration in (8.3) is easily performed, with the result

$$I_2(\zeta) = 1 + \frac{2}{\pi}\int_1^\infty \frac{\mathrm{d}t\; 2\epsilon\zeta\chi(t)}{t^2 + 4\epsilon^2\zeta^2\chi^2(t)} \tag{8.5}$$

(which is unchanged if the range of integration is replaced by $0 \leqslant t \leqslant 1$).

This integral was computed, and produced the graph of I_2 shown on figure 1. There is a weak maximum, receding to $\zeta = \infty$ as $D \to 1$, with coordinates

$$I_2 = 2{\cdot}211, \qquad \zeta = 1{\cdot}58/(D-1). \tag{8.6}$$

The decay from this maximum to saturation at $I_2 = 2$ is extraordinarily slow (as a function of $\epsilon\zeta$); it will soon be considered in analytical detail.

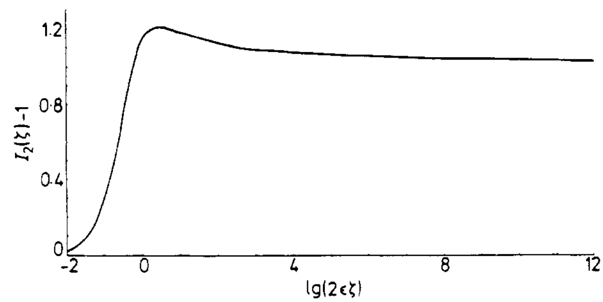

Figure 1. Second moment for the marginal diffractal $D = 1$ as a function of distance from the phase screen (computed by Dr J H Hannay).

When $\epsilon\zeta$ is small, the term in $\epsilon^2\zeta^2$ in (8.5) can be neglected, and the resulting integral of $\chi(t)/t^2$ performed analytically. In this way the limit (6.4) is regained.

Finally, I shall discuss the approach to the limit $\epsilon\zeta = \infty$. A treatment based on (8.5) is given in Appendix 2, but it is more instructive to use the more general result (7.14). This shows that as $D \to 1$ the number $N(D)$ of slowly decaying terms, given by (7.11), becomes infinite, and they all tend to $\zeta^{-2\epsilon(n+1)} \to 1$. To obtain the total contribution from this cluster of terms it is permissible to replace summation over n by integration over $\nu \equiv n+1$, so that (7.14) becomes

$$I_2 - 2 \to \frac{2}{\pi}\int_1^{1/2\epsilon} d\nu \frac{[4(1+\epsilon)(0{\cdot}5-\epsilon)]^\nu \Gamma(1-2\epsilon\nu)\sin\pi\epsilon\nu\, \Gamma[(1+\epsilon)\nu/(1-\epsilon)]}{\nu!\, 2^{(1+\epsilon)\nu/(1-\epsilon)}\zeta^{2\epsilon\nu}}$$

$$(\zeta \to \infty, \epsilon \to 0). \tag{8.7}$$

Use of Stirling's formula and expansion for small ϵ gives

$$I_2 - 2 \to \frac{2}{\pi}\int_{2\epsilon}^1 dx \frac{\sin\frac{1}{2}\pi x\,\Gamma(1-x)}{x} e^{x\ln x}\, e^{-x\ln(4\epsilon\zeta\, e^{3/2})}. \tag{8.8}$$

for large $\epsilon\zeta$ the integral is dominated by its lower limit, which may be set equal to zero, giving

$$I_2 \to 2 + 1/(\ln 2\epsilon\zeta + 2{\cdot}19) \qquad (\epsilon\zeta \to \infty). \tag{8.9}$$

This inverse logarithm is a surprisingly slow decay, whose origin in an accumulation of power laws is unprecedented in wave theory as far as I know. Its correctness is confirmed by the analysis of Appendix 2, and also by the computations leading to figure 1, which shows the tail of $I_2(\zeta)$ to be well fitted by

$$I_2 \to 2 + 0{\cdot}98/(\ln 2\epsilon\zeta + 4{\cdot}13), \tag{8.10}$$

whose leading terms agree closely with (8.9).

9. Conclusions

This study shows how rich in structure even the simplest diffractal can be. Since the evolution of a fractal wavefront appears to be a canonical mathematical problem, it is worth exploring further. In particular the following questions remain unanswered:

(i) How do the higher-intensity moments behave? Does the intensity have a stable, or an infinitely divisible, probability distribution (Jona-Lasinio 1975)?

(ii) How is the extreme fractal ($D = 2$) approached? A preliminary study, based on exact solution of the cut-off correlation

$$\langle (h(x+X)-h(x))^2 \rangle = L^2 \Theta(X-\delta), \tag{9.1}$$

where Θ denotes the unit step function, shows that as $\delta \to 0$ and the $D = 2$ fractal is reached (equation (2.5)) the second moment jumps rapidly from $I_2 = 1$ to saturation at $I_2 = 2$, in contrast to the marginal diffractal (§ 8) which saturates infinitely slowly. It is possible, however, that in the extreme fractal case the results might depend not only on D but also on the wavefront's Hausdorff measure, which is a more discriminating index of fractality.

(iii) How are wavefront dislocations (Nye and Berry 1974) distributed in a diffractal? Since the dislocations measure the disruption of wavefronts (Berry 1978), their changing density as z increases would be especially interesting for these fractal cases where the initial wavefront is as irregular as it can be.

(iv) Is it possible to find an exact solution for ψ when the phase screen is a non-random fractal, such as the Weierstrass function (Mandelbrot 1977, Berry and Lewis 1979)?

(v) What about non-monochromatic diffractals? The extreme case is the reflection of a delta-function pulse from a fractal surface, and preliminary arguments (Berry 1972) indicate that the wave as a function of time is singular everywhere and may itself be a fractal.

(vi) How is the spectrum of a spatially incoherent wave altered by encounter with a fractal screen?

Beyond the fractal phase screen model lies a whole realm of more difficult diffractal problems. Some of these concern scattering from fractals that are not functions, like the Siepinski sponge (Mandelbrot 1977), which could represent porous (oil-bearing?) rock being probed by echo sounding. Then there are eigenvalue problems: what are the solutions of Schrödinger's equation in a Weierstrass potential? What are the high harmonics of a drum shaped like the Koch snowflake curve (Mandelbrot 1977)†? I do not know how far the results of this paper will help in understanding these more complicated diffractals.

Acknowledgments

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Appendix 1. Fractal dimension of Gauss random functions

Following Orey (1970) I use the 'potential' definition of fractal dimension (Mandelbrot 1977). Let positive charge uniformly cover the region $-l/2 < x < l/2$ of the x axis with unit density. Move this charge up or down in the h direction until it hits the graph of

†*Note added in proof.* I shall publish a conjecture about the asymptotic mode distribution in fractal resonators in Berry (1979).

$h(x)$. Now write the electrostatic energy of this fractal line of charge, employing a modified Coulomb law where the interaction potential of two unit charges separated by distance r is $r^{-\mu}$, where μ need not be unity. This energy is $E(\mu)$, where

$$E(\mu)=\frac{1}{2}\int_{-l/2}^{l/2}\mathrm{d}x\int_{-l/2}^{l/2}\mathrm{d}x'[(h(x)-h(x'))^2+(x-x')^2]^{-\mu/2}. \tag{A1.1}$$

Then D is defined as the greatest μ for which $E(\mu)$ is not infinite.

To study the convergence of (A.1.1) attention will be restricted to 'almost all' functions h in the Gaussian ensemble by evaluating the average $\langle E(\mu)\rangle$. Denoting $h(x)-h(x')$ by Δ, and its probability distribution by $P(\Delta)$, and employing $X\equiv x-x'$ as a new variable, the average energy becomes

$$\langle E(\mu)\rangle=\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}\Delta\,P(\Delta)\int_{-l}^{l}\mathrm{d}X(\Delta^2+X^2)^{-\mu/2}(l-|X|). \tag{A1.2}$$

From (2.1), (2.5) and (2.6),

$$P(\Delta)=\mathrm{e}^{-\Delta^2/2L^{2D-2}X^{4-2D}}/\sqrt{2\pi}L^{D-1}X^{2-D}. \tag{A1.3}$$

Convergence depends on the behaviour at $\Delta=X=0$, so all constants can be set equal to unity and the integral under study written as

$$\langle E(\mu)\rangle\propto\int_{-\infty}^{\infty}\mathrm{d}\Delta\int_{-\infty}^{\infty}\mathrm{d}X\frac{\mathrm{e}^{-\Delta^2/X^{4-2D}}}{X^{2-D}(\Delta^2+X^2)^{\mu/2}}. \tag{A1.4}$$

With the polar coordinates r, θ in the Δ, X plane this becomes

$$\langle E(\mu)\rangle\propto\int_0^{2\pi}\mathrm{d}\theta\int_0^{\infty}\mathrm{d}r\frac{r^{D-1-\mu}\,\mathrm{e}^{-r^{2(D-1)}/(\sin\theta)^{2(2-D)}}}{(\sin\theta)^{2-D}}. \tag{A1.5}$$

As $r\to 0$ the exponential factor is unity except in a narrow sector of angular width $\theta\propto r^{(D-1)/(2-D)}$. The singularity $(\sin\theta)^{-(2-D)}$ is integrable, so convergence is governed by the factor $r^{D-1-\mu}$ and requires

$$\mu<D. \tag{A1.6}$$

Therefore the dimension of the graph of $h(x)$ is D. (Q.E.D.)

Appendix 2. Asymptotic behaviour of the marginal diffractal

The naive approach to finding I_2 from (8.5) for large $\epsilon\zeta$ is to formally expand in inverse powers. However, this gives for the first term the integral of χ^{-1}, which by (8.4) diverges for large t. Therefore large t dominates the integral in (8.5) when $\epsilon\zeta$ is large, and I_2 is given by

$$I_2(\zeta)\approx 1+\frac{4(2\epsilon\zeta)}{\pi}\int_1^{\infty}\frac{\mathrm{d}t\ln t}{t^2+(4\epsilon\zeta\ln t)^2}\qquad(\epsilon\zeta\gg 1), \tag{A2.1}$$

whose asymptotic behaviour is to be determined.

This is quite subtle. The most efficient procedure seems to be to start with the expansion in ascending powers of $2\epsilon\zeta$, namely

$$I_2(\zeta)-1=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(4\epsilon\zeta)^{2n+1}(-1)^n(2n)!}{(2n+1)^{2n+1}}, \tag{A2.2}$$

and transform to a Mellin representation as explained by Dingle (1973). This gives

$$I_2(\zeta)-1=\frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\frac{\mathrm{d}u(-1-u)!(-u)^u\,\mathrm{e}^{-u\ln(4\epsilon\zeta)}}{\cos\frac{1}{2}\pi u}, \tag{A2.3}$$

where $-1<\gamma<0$. To recapture (A2.2), simply shift the contour towards Re $u=-\infty$ and collect the poles of $(\cos\frac{1}{2}\pi u)^{-1}$.

The required descending expansion comes from the other singularities of the integrand. These are simple poles at $u=+2n$ from the factorial, double poles at $u=2n+1$ from the cosine and the factorial, and a branch point at $u=0$ from the multivalued function $(-u)^u$. The simple poles, apart from that at the origin, giontributions of order $(\epsilon\zeta)^{-2n}$, and the double poles give contributions of order $\ln(\epsilon\zeta)/(\epsilon\zeta)^{2n+1}$. What is of interest here, however, is the dominant contribution, which comes from the origin. Looping the contour in (A2.3) to surround the branch cut, which will be taken along the positive imaginary axis $u=\mathrm{i}y$, and separating the pole and cut contributions gives

$$I_2-1=1+\frac{2\mathrm{i}}{\pi}\int_0^{\infty}\mathrm{d}y\frac{(-\mathrm{i}y)!\sinh\frac{1}{2}\pi y}{y}\,\mathrm{e}^{-\pi y/2}\,\mathrm{e}^{\mathrm{i}y\ln y}\,\mathrm{e}^{-\mathrm{i}y\ln(4\epsilon\zeta)}+O\left(\frac{\ln\epsilon\zeta}{\epsilon\zeta}\right), \tag{A2.4}$$

where the correction term comes from the poles along the positive real axis. When $\epsilon\zeta$ is large, the integral (which closely resembles (8.8)) is dominated by small y, so that

$$I_2\to 2+1/(\ln 2\epsilon\zeta+0{\cdot}69), \tag{A2.5}$$

in agreement with (8.9) up to leading terms.

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